

Monoidal category of operad of graphs

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ABSTRACT. Usually a name of the category is inherited from the name of objects. However more relevant for a category of objects and morphisms is an algebra of morphisms. Therefore we prefer to say a category of *graphs* if every morphism is a graph.

In a monoidal category every morphism can be seen as a graph, and a partial algebra of morphisms possesses a structure of an operad, operad of graphs. We consider a monoidal category of operad of graphs with underlying graphical calculus. If, in particular, there is a single generating objects, then each morphism is a binary graph. The graphical calculus, multi-grafting of morphisms, is developed *ab ovo*.

We interpret algebraic logic and predicate calculus within a monoidal category of operad of graphs, and this leads to the graphical logic.

A logic based on a braided monoidal category is said to be the braided logic. We consider a braided monoidal category generated by one object. We are demonstrating how the braided logic is related to implicative algebra and to the Heyting algebra (in contrary to the Boolean algebra) and therefore must be more related to the quasigroups than to the lattices.

Some applications to classical logic, to modal logic and to Łukasiewicz three-valued logic are considered.

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1. Operad of graphs

We work within a monoidal category (almost a tensor category) generated by a single object. In this case the set of all objects of the single generated monoidal category coincides with the set of non-negative integers (with the set of natural numbers) \mathbb{N} . Thus the set of all objects is, $\text{obj } \mathbf{cat} = \mathbb{N}$, and $1 \in \mathbb{N}$ is a generating object.

Therefore each morphism (an arrow of a category) is characterized by a *pair* of non-negative integers, morphisms are bi-graded, and we refer to this pair $\{\text{input}, \text{output}\} = \{\text{entrance}, \text{exit}\}$, as to the type or arity of the morphism $= \{\text{arity-in}, \text{arity-out}\}$,

$$(1) \quad \mathbb{N} \ni m \xrightarrow{\text{morphisms of } (m \rightarrow n)\text{-arity}} n \in \mathbb{N}.$$

Garrett Birkhoff in his *Lattice Theory* in 1940, within Universal algebra considered an n -ary operation symbol to be carefully distinguished from a model of the operation. Traditionally, every operation symbol in an algebra is of arity (of type) $\in \mathbb{N}$, *i.e.* of arity $\mathbb{N} \ni n \mapsto 1$ with one exit. Every such operation is considered here to be a morphism in a monoidal category. For example the traditional 0-ary (or null-ary) operation in our terminology must be bi-graded. It is a $(0 \rightarrow 1)$ -operation/morphism, $\in \mathbf{cat}(0, 1)$ Similarly traditional n -ary operations are seen here as morphisms $\in \mathbf{cat}(n, 1)$.

We need to introduce the graphical notation we are using. Every morphism $\in \mathbf{cat}(m, n)$, $m, n \in \mathbb{N}$, is visualized as a node with a number of outer leaves (representing the source and target objects): on top the

input object of m -arity-in, and at bottom the output object of n -arity-out as illustrated in Figure 1. Throughout this paper the graphs are directed and we read them from the top to the bottom. Bi-gradation is in accordance with the time arrow. Reading these graphs upside down, from the bottom to the top, can be equivalent to the inverse of the time's arrow.

For example, a morphism $\in \mathbf{cat}(2, 1)$ in Figure 1 is traditionally called a binary algebra. Each morphism is a graph with no outer nodes. Every node is inner, and every edge is an outer leave or an inner edge, which represents an object of a monoidal category.

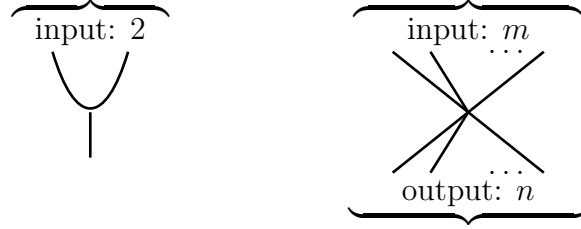


FIGURE 1. A $(2 \rightarrow 1)$ -morphism $\in \mathbf{cat}(2, 1)$, and $(m \rightarrow n)$ -morphism $\in \mathbf{cat}(m, n)$ is an arrow from m to n .

Two morphisms with the same arity-in and arity-out are said to be *parallel*, they are parallel arrows, $m \rightrightarrows n$.

1.1. Definition (Schizophrenic). In monoidal category \mathbf{cat} , we postulate a set, $\Omega \equiv \mathbf{cat}(0, 0)$, to be a candidate for a dualizing or schizophrenic set. A plant $0 \mapsto 0$ can be always blot out.

Morphisms can be composed in many different ways, like garden plants are grafted. Composition of morphisms generalizes the construction of words from an alphabet, however words are constructed by concatenations only, whereas our bi-graded morphisms allow diverse ‘non-linear’ compositions.

We refer to each composition as a grafting. The concatenation or juxtaposition of arrows is considered also as a special grafting. For example a concatenation of an arrow from $\mathbf{cat}(2, 1)$ with another arrow from $\mathbf{cat}(2, 1)$, gives an arrow from $\mathbf{cat}(4, 2)$, see Figure 2.

Occasionally a bigraded morphism we will call in a diverse way, a graph, a garden *plant* with root and crown of branches, operation, symbol, letter, atom, a formal predicate. The multitude of names is better expressing what is going on.

Except for concatenations that will be not used in what follows, each grafting is also bigraded, and $(i \rightarrow j)$ -graft means joining of the

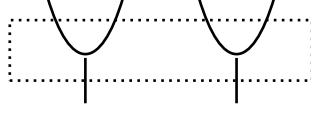


FIGURE 2. The concatenation of $Y \in \mathbf{cat}(2, 1)$ with Y gives an arrow in $\mathbf{cat}(4, 2)$.

i -th output of the first $(m \rightarrow n \neq 0)$ -morphism, $i \in \{1, 2, \dots, n\}$, with j -th input of the second $(0 \neq k \rightarrow l)$ -morphism, $j \in \{1, 2, \dots, k\}$, as illustrated in several examples in next Figures.

‘The plants’ can be grafted in more ways than ‘the letters’ can be juxtapositioned. Reader can freely choose the favorite name.

1.2. Definition (Free operad). Every chosen finite set of such bi-graded morphisms we treat as an alphabet generating, by all possible graftings, a free operad (clon, abstract algebra, variety). Every generator of an operad is of the type in: entrance $\mathbb{N} \ni m \mapsto n \in \mathbb{N}$ exit, *i.e.* all plants are bi-graded as it is the case in the nature.

A null-ary operation, a creator, is of arity $(0 \mapsto 1) \simeq \mathbf{cat}(0, 1)$. The null-ary co-operation, a killer, is of type $(1 \mapsto 0) \simeq \mathbf{cat}(1, 0)$, and is unique, so that a category possess the terminal object. The killer is the process of forgetting all, the annihilator. An un-ary cooperation or operation is of type $(1 \mapsto 1) \simeq \mathbf{cat}(1, 1)$, and the general type is: entrance $\mathbb{N} \ni m \mapsto n \in \mathbb{N}$ exit.

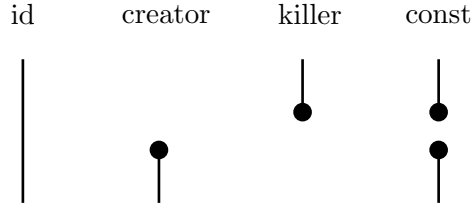


FIGURE 3. From the left: $\text{id} \simeq | \in \mathbf{cat}(1, 1)$, $\text{creator} \in \mathbf{cat}(0, 1)$, $\text{forgetting} \in \mathbf{cat}(1, 0)$, and composed the constant morphism $\in \mathbf{cat}(1, 1)$.

1.3. Definition (Quotient operad). An equivalence relations among parallel morphisms, among parallel arrows, determine two-sided ideal in the free operad. This leads to the quotient operad. On Figure 4, three identities (the equivalence relations in an operad among parallel morphisms) on the right are, by definition, the abstract minimum polynomials.

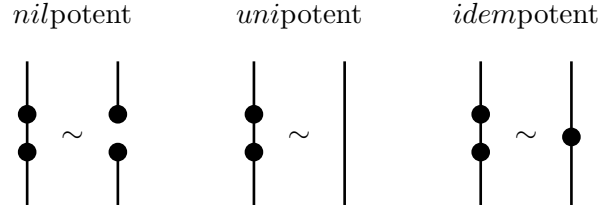


FIGURE 4. Three examples of unaries $\in \mathbf{cat}(1,1)$ defined by relations among parallel morphisms.

The boolean negation is an example of the unipotent unary operation, but nil- and idem-potents are not classical, *i.e.* does not exist for the two elements object $\{\text{true}, \text{false}\}$.

Joyal & Street [1991] introduced the *valuation* of a graph as the pair of applications [Joyal& Street 1991, Definition 1.3, p. 64],

$$\text{edges} \longrightarrow \text{obj } \mathbf{cat}, \quad \text{nodes} \longrightarrow \text{arrows } \mathbf{cat}.$$

In our convention graph has no outer nodes. Every node is inner and represent a functor or a map, *i.e.* a process, an action, a co-action, operation, multi-functor, evaluation, function, etc. Every edge, including outer leaves, represent object.

A bifunctor of bin-ary operation, an anihilation $\in (2 \mapsto 1) \simeq \mathbf{cat}(2,1)$, denoted by two initial leaves and one node. A binary co-operation is a decomposition (of information), a splitting, duplication with the mutants, procreation process $\in (1 \rightarrow 2) \simeq \mathbf{cat}(1,2)$. The bin-aries: anihilation $\in (2 \mapsto 1) \simeq \mathbf{cat}(2,1)$, creation $\in (1 \mapsto 2) \simeq \mathbf{cat}(1,2)$, and scattering $\in (2 \mapsto 2) \simeq \mathbf{cat}(2,2)$, are represented by the prime graph nodes on Figure 5.

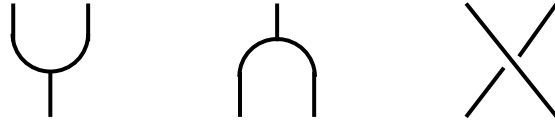


FIGURE 5. The binaries: operation (anihilation), co-product (procreation) and a scattering (prebraid).

1.4. Definition (The plication). A pro-creation process $\in (1 \mapsto n \geq 2) \simeq \mathbf{cat}(1,n)$, in realization the process of the copy of an identical variables, informations, thought, ideas, things, species, genres, ... [Reader can freely choose the favorite name], is called n -plication (duplication, tri-plication, multi-plication) or n -ary plication. The (du)-plication is co-associative and co-commutative, is the ‘group-like’ co-operation like reproduction in biology, like mitosis, cariocinesis $\Delta : a \mapsto$

(a, a) . Compare with [Lambek & Scott, page 8, Exercise 3]. Abstractly (two-sided) duplication is defined by the pair of relations on Figure 6.



FIGURE 6. Mitosis.

1.5. Comment. In realization on the set S , mitosis is known as the diagonal or identity relation on S , mitosis $\in 2^{(S \times S)}$. The name *multiplication* we use for the co-process $a \mapsto (a, a, \dots, a)$, contrary to the usual meaning in literature of the binary operation.

Every binary cooperation is represented diagrammatically by node with two exit edges (two exit streams) as shown on Figures 6 and 7.

Every general co-operation $\in \mathbf{cat}(1, n)$, like meiosis in biology $\in \mathbf{cat}(1, 4)$, can be considered as the composition of the multiple mitosis with a set of n (different) modal unary operations which we interpret as the *mutant* operations for cooperations.



FIGURE 7. Every cooperation (read from the top) is the grafting (composition) of the duplication with the pair of unary mutants.



FIGURE 8. A right mutant from binary operation.

1.6. Lemma. *Let a and b be unaries (modal). Then binary co-operation (a, b) is coassociative iff a and b are commuting idempotents.*

Composition of the ‘elementary’ plants from Figures 3 and 5 looks like the grafting in an orchard and this grafting is generating the operads of composed plants. For example, each of the two grafted plants $\in \mathbf{cat}(4, 2)$ and $\in \mathbf{cat}(2, 4)$ on Figure 12 is the results of the grafting of three elementary colored plants from Figure 5.

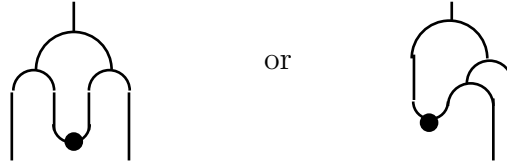


FIGURE 9. .



FIGURE 10. Co-associativity of binary procreation.

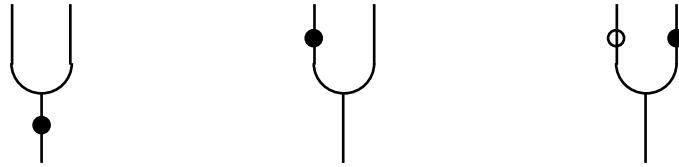
FIGURE 11. Examples of $2 \mapsto 1$ grafted with $1 \mapsto 1$.

FIGURE 12. Product of binary operations and of co-operations, needs the grafting of three plants including braid.

2. Clone = operad of plants. Hypervariety, Bootstrap

The grafting of plants is our basic ‘operation’. Grafting is multi-valued, therefore is not an operation in the usual meaning. From two copies of binary plant grafting produce two different ternaries, etc. We prefer use ‘plant’ instead of ‘operation’ because a realization in sets is not yet assumed (not yet carriers). An algebra in universal algebra is a set (carrier of algebra) with a family of operations. Instead, in this paper, we deal with the family of plants of the given type $\langle \dots \rangle$ - a generator of a free operad - and with relations, however carrier was not yet selected. Therefore by an algebra we mean a family of plants of the given type $\langle \dots \rangle$.

We identify variety of algebras with the quotient operad.

Contrary to distinguished unique co-operation of duplication $a \mapsto (a, a)$ (Figure 6) there are no distinguished binary operation. Figure 6 readed from the bottom define binary with neutral. Not every binary operation can be constructed in terms of some distinguished binary and the set of all modal operations.

2.1. Exercise. For the case of the two element set $\{\text{true}, \text{false}\}$, the classical clone is one generated clone by a $\{\text{Sheffer stroke}\}$,

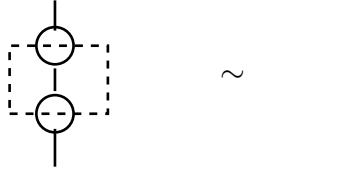


FIGURE 13. The Sheffer stroke.

Above we defined an operad on generators (alphabet of morphisms) and relations. This was presentation-dependent quotient operad. One can try to define a clone without of presentation, as presentation-free operad by means of the axiomatic abstract properties of the given operad. We are not going to this subject in the present paper.

3. The Boolean and Heyting operations

The *logic* in the present paper is an abbreviation for the algebraic categorical logic. For example the modal logic needs full set of unary operations and their mutual inter-relations.

We define the Boolean operation and the Heyting operation [Heyting 1930, Henkin 1950] as the relation on two nodes graph with one bubble, as shown on Figure 14. The Heyting operation is an algebra generated by two plants of type $\in \mathbf{cat}(2, 1)$ and $\in \mathbf{cat}(0, 1)$ with this one relation. For example, the Heyting operation enter into BCC-algebra and BCK-algebra invented as the models for implicational propositional calculus [Iséki 1966].

The first node $\in \mathbf{cat}(1, 2)$ represents either the pure duplication or a duplication with essential mutants (an emission, pro-creation, decomposition of information, etc) and the next node $\in \mathbf{cat}(2, 1)$ represents an essential binary (primitive, *i.e.* not unary with killer), absorption, annihilation, consumption, reinforcement, \dots

We do not assume that the Boolean and the Heyting binaries need to be associative and/or commutative. If instead for some finite integer $n > 1$ the (abstract) minimum polynomial of this bubble is $B^n = \text{id}$



FIGURE 14. Two semi-lattices: the Heyting semi-lattice and the Boolean semi-lattice.

(i.e. the bubble B is neither Boolean, nor the Heyting), then is said to be (generalized) Shefferian, Pierce, or Post operation.

The Heyting operation generalize an implicative algebra [Rasiowa 1974, p. 16]. An implicative algebra play for some non-classical logics a role analogous to that played by the Boolean algebra for the classical logic.

For the Boolean semilattice, the Heyting semilattice, the Boolean \mathbb{Z}_2 -algebra, the Heyting algebra, we refer to [Lambek Scott, p. 36, Examples 7.3-7.4].

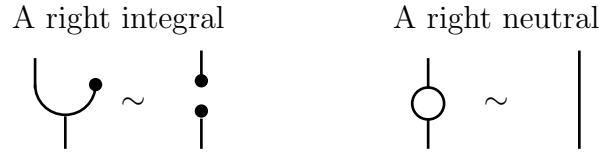


FIGURE 15. Integral and neutral.

4. Lattice and Quasi-Group

The filled and not filled circles on Figure 16 represent two binaries. At this general setting neither associativity nor commutativity need to be assumed. We are contrasting the interrelations between two binaries defining the (right) lattice (where the matter survive) and the (right) quasigroup (where matter is cancelled [Moufang 1935]). Both relations (equations, identities) on Figure 16 define *strongly nonregular* clones (or varieties) considered by Graczyńska [1989, 1990, 1998, 2006]. Terminology *regular* clone, etc, was introduced by [Płonka 1969].

5. Braid

An endomap of arity (type) $\in \mathbf{cat}(2, 2)$, is said to be *pre-braid* if the Artin prebraid relation $\in \mathbf{cat}(3, 3)$, represented by tangles on Figure 17, holds. An invertible prebraid is said to be a braid.

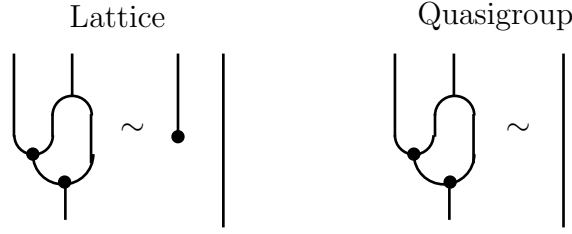


FIGURE 16. The right lattice and the right quasigroup (the right cancellation).

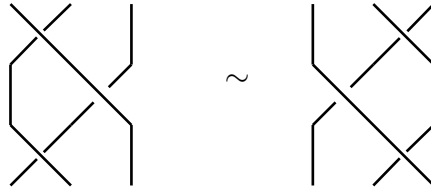


FIGURE 17. The Artin prebraid relation.

Let $\text{plant} \in \mathbf{cat}(2,2)$ is composed from a pair of from $\alpha, \beta \in \mathbf{cat}(2,1)$. Therefore we can insert into every vertex on Figure 17 the brassiere Figure 18, *i.e.* a pair (α, β) of the binary operations



FIGURE 18. Brassiere.

Then it is easily seen that the Artin relation on $3 \mapsto 3$ is equivalent to three ternary (regular? [Płonka]) relations shown on Figure 19.

5.1. Definition (Essential morphism). All operations $\in \mathbf{cat}(m,1)$ (and also all plants $\in \mathbf{cat}(m,n)$) are segregated into two groups: not essential (or trivial) and essential or primitives (not trivials). An unary operation $\in \mathbf{cat}(1,1)$ is said to trivial if it is a constant map (the composition of the killer with nullary operation) or the identity plant. All other unaries are said to be essentials.

For example in realization on the finite set S with the cardinality $s \equiv |S|$, the set of all unaries has the cardinality s^s . The number of

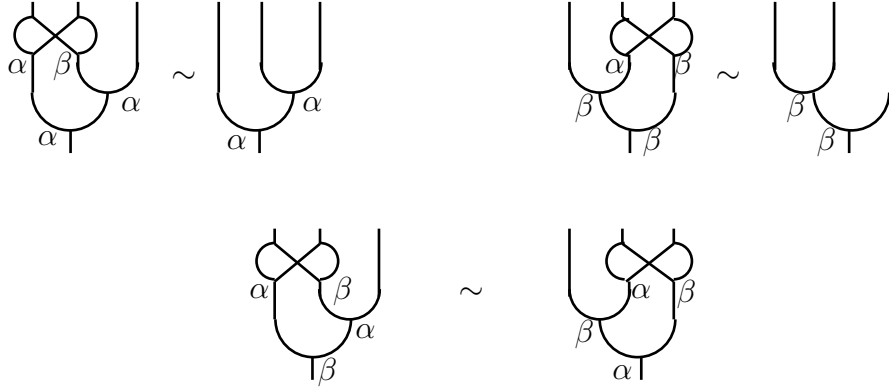


FIGURE 19. A pair of binary operation as the Artin prebraid.

primitive unaries is

$$s^s - s - 1 = \begin{cases} 1 & \text{if } s = 2 \text{ i.e. in classical logic,} \\ 23 & \text{if } s = 3 \text{ in the Łukasiewicz logic.} \end{cases}$$

A plant $1 < m \mapsto 1$ is said to be derived (or grafted, not essential, trivial) if can be build by juxtapositioning and grafting from plants $m > k \mapsto 1$. For example the number of the essential binaries $2 \mapsto 1$ in the realization on the finite set S as above, is

$$s^{(s^2)} - 2s^s + s = \begin{cases} 2^4 - 6 = 10 & \text{in the classical case,} \\ 3^9 - 51 = 19632 & \text{in the Łukasiewicz logic.} \end{cases}$$

Therefore among 16 binary classical connectives, 6 are not primitives because they are juxtaposition of unary with killer, and 10 are essential. In particular ‘projector’ $m \mapsto 1$ is juxtaposition of the killers with the identity plant.

Our first result concern the classical two-valued logic. In this case a priori there are 100 pairs of primitives binaries as the candidates for the Artin prebraid.

5.2. Corollary. *Let in Figure 19 the both binary plants be primitive. Then in the classical (two-valued logic) there are four prebraids build from the disjunction and the conjunction only. All these prebraids are idempotents.*

5.3. Corollary. *In the classical logic both the disjunction and the conjunction are (α, β) -symmetric.*

PROOF. This statement is equivalent to the idempotency of the pre-braids. \square

If in a plant $\in \mathbf{cat}(2, 2)$, $\alpha \equiv \beta$, then brassiere is reduced to the duplication of one binary as shown on the first graph in Figure 19. Then every boolean operation (necessarily associative) gives prebraid.

The other four graphs on Figure 19 correspond to the cases when at least one among two binaries in the brassiere is given by unary (with killer). In these cases a binary cooperation needs not to be just duplication.

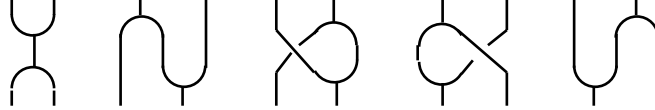


FIGURE 20. Examples of grafted $\in \mathbf{cat}(2, 2)$.

5.4. Conjecture. *If at least one binary among (α, β) in Figure 19 is unary (with killer) then the last Artin relation in Figure 19 is equivalent that $\alpha \in \text{hom } \beta$ or $\beta \in \text{hom } \alpha$.*

Figure 21 gives the two examples of the ‘identities’, two examples of the possible laws of the nature. Two plants grafted on the left of each law are identical as three plants grafted on the right. These laws are interpreted that a binary $\in \mathbf{cat}(2, 1)$ (an operation, an action, et cetera) is the morphism with respect to the process $\in \mathbf{cat}(2, 2)$, the left law is *under* morphism, the right law is *over* morphism. In particular Figure 17 is nothing but morphism (simultaneously under *and* over) of a prebraid $\in \mathbf{cat}(2, 2)$ with respect to himself, *i.e.* selfmorphism.

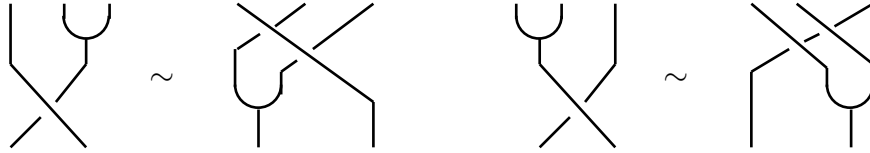


FIGURE 21. A binary as the morphism: under and over.

Specializing Figure 21 to the relations expressing that binary γ is the (α, β) -morphism we get the system of (regular?) relations on Figures 22 and 23.

The relations on the left in Figures 22 and 23 coincide with the so called condition (S) in BCK-algebra of a type

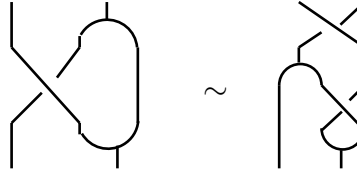
$$(2) \quad \langle 2, 2, 0 \rangle \simeq \mathbf{cat}(2, 1), \mathbf{cat}(2, 1), \mathbf{cat}(0, 1).$$

5.5. Theorem. *Let α and β be two Heyting binaries and (α, β) -morphisms. Then a clon (of abstract algebras) $(\alpha, \beta, 0, 1)$ of type $(2, 2, 0, 0)$ is quasigroup.*

FIGURE 22. Binary γ is under (α, β) -morphism.FIGURE 23. Binary γ is over (α, β) -morphism.

PROOF. In Figures 22 and 23 we must take $\gamma = \alpha$ and $\gamma = \beta$ and graft the duplication on the tops. Then insert the Heyting operation from Figure 13 with two nullary operations. \square

If a plant of cooperation is under σ -morphism then the tangles in Figure 21 can be reexpressed as braided plant as shown on Figure 24.

FIGURE 24. Under σ -morphism.

6. Cooperation as the morphism in a braided category

When we wish to include an cooperation from Figure 7 as the morphism in the prebraided monoidal category generated by prebraid (α, β) , then this leads naturally to new two candidates for the Artin prebraids, so to say we get, ‘the system’ of prebraids. Two new processes $\in \mathbf{cat}(2, 2)$ which enter into the game are given on Figure 25.

In the sequel we abbreviate for short,

$$(3) \quad (\heartsuit, \alpha) \equiv (\text{id} \times \alpha) \circ (\heartsuit \times \text{id}),$$

$$(4) \quad (\beta, \heartsuit) \equiv (\beta \times \text{id}) \circ (\text{id} \times \heartsuit).$$

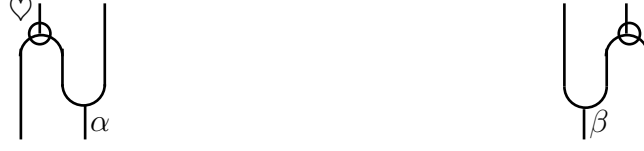


FIGURE 25. Growing system of braids.

6.1. Theorem. *Let a and b be unaries. Then the cooperation $\heartsuit \equiv (a, b)$ is under (α, β) -morphism iff*

- (i) β must be (β, \heartsuit) -symmetric.
- (ii) $\alpha \in \text{hom}((\beta, \heartsuit), b)$.
- (iii) $\alpha \in \text{hom}(\text{id} \times a, a)$.

The cooperation \heartsuit is over (α, β) -morphism iff

- (i) α must be (\heartsuit, α) -symmetric.
- (ii) $\beta \in \text{hom}((\heartsuit, \alpha), a)$.
- (iii) $\beta \in \text{hom}(b \times \text{id}, b)$.

7. Bigebra

The Boolean cogebras in combinatorics has been considered by Joni & Rota [1979]. However we believe that what is most relevant to the logic is the bi-gebra as in Figure 26 as the morphism, and in particular the Boolean bigebra. A general bigebra is a triple: a pre-braid of type $\in \mathbf{cat}(2, 2)$, binary $\in \mathbf{cat}(2, 1)$ and cooperation $\in \mathbf{cat}(1, 2) \simeq \mathbf{cat}(1, 1) \times \mathbf{cat}(1, 1)$.

Therefore a bigebra has the type

$$(5) \quad < 2, 2, 2, 1, 1 > \\ \simeq \mathbf{cat}(2, 1) \times \mathbf{cat}(2, 1) \times \mathbf{cat}(2, 1) \times \mathbf{cat}(1, 1) \times \mathbf{cat}(1, 1).$$

with at least two defining relations including the Artin relation and Figure 26.

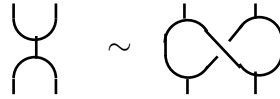


FIGURE 26. A bi-gebra with one pre-braid.

For the cloning (mitosis) and for the switch, Figure 26 holds for every binary operation, *i.e.* this bi-operation is always present implicitly. We wish to use this relation and generalize explicitly.

8. Dualizing object

For duality theory for Boolean algebras see [Stone 1936, Yetter 1990, Davey & Priestley 1990, Davey 1993].

8.1. Exercise (Davey 1993, p. 105, Problem 3). Which finite algebras admit a duality?

Let $\Omega, A \in \mathbf{objcat}$. An object Ω is said to be dualizing (or schizophrenic) if for every object A , there is an isomorphism between A and Ω^A . In this case an object Ω^A is said to be the dual to A . In the categories of the finite algebras Ω is dualizing iff $|\Omega^A| = |A|$.

8.2. Exercise. A category of the finite sets with one modal structure. Let $a \in A^A$ and $z \in \Omega^\Omega$ be modal (unary) structures,

$$(A, a), (\Omega, z) \in \mathbf{objcat},$$

$$f \in (\Omega, z)^{(A, a)} \iff f \circ a = z \circ f.$$

Determine $|(\Omega, z)^{(A, a)}| =$

If invariant subset $\text{inv}_a \equiv \{x \in A | ax = x\}$ is not empty, and inv_z is empty then $|(\Omega, z)^{(A, a)}| = \emptyset$.

In particular let all orbits of modal structures be of cardinality 2 (then modals are unipotents and cardinality of sets must be even). If $|\Omega| = 2$ and $|A| = 2n$, then $|(\Omega, z)^{(A, a)}| = 2^n$.

More complicated structure of orbits? See [Stone 1936].

8.3. Exercise. A category of the finite sets with binaries. Let

$$\alpha \in A^{(A \times A)}, \quad \omega \in \Omega^{(\Omega \times \Omega)}.$$

Let C be co-magma and A be magma. Then A^C inherit a structure of magma with a convolution product. If magma A is finite then C^A inherit a structure of comagma. If $|A| < \infty$ and Ω is a dualizing object then $A^* \equiv \Omega^A$ is comagma and this is displayed on Figure 27.



FIGURE 27. The product - co-product duality.

We need first to have dualizing object $\Omega = \mathbf{cat}(0, 0)$. Maybe we must allow that $|\Omega^A| > |A|$?, in order to have the perfect duality (*i.e.* bijection) in binary operations? That is to every binary on A corresponds just unique cobinary on $|\Omega^A|$?

9. The Wigner & Eckart problem: the Clebsch-Gordan coefficients

A left action of a bi-magma or a bi-monoido \mathcal{M} is represented on Figure 28: the binary tree on the left is an action on a single \mathcal{M} -set and the next tangle represent the action of \mathcal{M} on cartesian (or tensor) product of two \mathcal{M} -sets. This last action depends on co-product.



FIGURE 28. An action of a bi-magma \mathcal{M} on left \mathcal{M} -set and a co-product dependent action of \mathcal{M} on the cartesian product of two left \mathcal{M} -sets.

Every automorphism (a permutation) $\alpha \in \text{aut } \mathcal{M}$ of a magma, give rise to the another action $\gamma \rightarrow (\alpha \times \text{id}_S) \circ \gamma$. Two actions γ and γ' are equivalent if exists a bijection φ such that $\varphi \circ \gamma = \gamma' \circ (\text{id}_{\mathcal{M}} \times \varphi)$.

The tangle on Figure 28 is the *definition* of the (cartesian) product of two \mathcal{M} -sets resulting again in a \mathcal{M} -set. Therefore the product of two \mathcal{M} -sets depends on a pre-braid and is defined in terms of a coproduct on Figures 7 and 28.

An invariant operator T (in linear algebra known as a tensor operator) with respect to bi-magma \mathcal{M} is defined as a bin-ary morphism of left \mathcal{M} -sets and this definition is shown on Figure 29 . In particular T is a \mathcal{M} -invariant (commuting with the action of \mathcal{M}) binary operation of one-sided \mathcal{M} -sets.

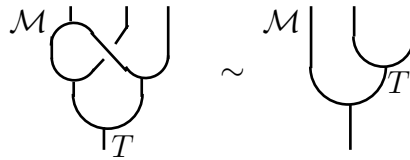


FIGURE 29. The morphism T of left \mathcal{M} -sets: T is \mathcal{M} -invariant.

The invariant operator T is operating on (an abelian?) category of all (one-sided) \mathcal{M} -sets. A category of \mathcal{M} -sets together with a binary morphism T is \mathcal{M} -set binary-algebra. Figure 29 has the three other interpretations:

Imprimitivity: A system of imprimitivity [Weyl 1931]. This is the case if \mathcal{M} is a group and if T is a ‘projection-valued measure’.

Measuring: An action of \mathcal{M} is measuring bin-ary multiplication T .

Distributing: An action of \mathcal{M} distribute T . If an action of \mathcal{M} is denoted by dot \cdot and $T = +$, then Figure 29 tells,

$$\forall c \in \mathcal{M}, \quad (c \cdot m) + (c \cdot n) = c \cdot (m + n).$$

Evaluation: For example in $*$ -autonomous category.

A \mathcal{M} -invariant binary T depends on a co-product Δ . An open problem is the explicit determination of the invariant operators T for the given bimagma \mathcal{M} .

10. Realizations. Models in sets. Representation

If A, B, C are objects in cartesian closed category then

$$(6) \quad (A \times B)^C = (A^C) \times (B^C), \quad A^{(B \times C)} = (A^B)^C.$$

Throughout this paper all sets are finite. Let S be nonempty set, $s \equiv |S| \in \mathbb{N}$. The multiple cartesian products are denoted as

$$S^{\times n} \equiv \underbrace{S \times S \times \dots \times S}_{n \text{ times}},$$

with $S^{\times 0} \equiv$ one element set, $S^{\times 1} \equiv S$, $S^{\times 2} \equiv S \times S$, etc.

A clone (language, operad) of operations (graphs, plants) of type $m \mapsto n$ can be realized (a model) inside of the power sets

$$(7) \quad \text{clone} \longrightarrow (S^{\times n})^{(S^{\times m})}.$$

Representation of a clone is a contravariant functor. If map (7) is injective than a set S is said to be a carrier for a clone.

An endomap of type $2 \mapsto 2$, can be realized for example as a map $\sigma : S \times S \longrightarrow S \times S$, $\sigma \in \text{End}_{\mathcal{F}}(S \times S)$ is said to be *pre-braid* if σ solve the Artin prebraid relation $3 \mapsto 3$ represented by tangles on Figure 17.

11. The Łukasiewicz logic

The three valued logic $\{\text{true}, \text{undefined}, \text{false}\}$, [Łukasiewicz 1918, Post 1921, Kleene 1952].

TABLE 1. Some modal (unary) operations among 23 primitives

	true	undefined	false	minimum polynomial	
possibility	t	t	f	$P^2 = P$	Łukasiewicz
necessity	t	f	f	$N^2 = N$	Łukasiewicz
contingent	f	t	f	$C^2 = \text{const}$	
rotation	u	f	t	$R^3 = \text{id}$	Post 1920
pseudo	f	f	t	$H^3 = H$	Heyting 1930, 1966
negation	f	u	t	unipotent	Kleene 1938
knowledge					Kleene 1952
belief					Kleene 1952

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